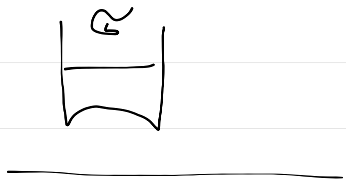
Recap Following Borel's exposition, we considered (for $G = SL_2$) the system of equations, on moderate growth functions φ on $\Gamma \backslash G$ of (given K-type m),

$$(A_1) \quad \varphi_P\big|_{\mathfrak{S}} = f_{P,s} + c'(s) f_{P,-s} \qquad P = \begin{pmatrix} * & * \\ & * \end{pmatrix}$$

for some $c'(s)$.



truncation operator $\Leftrightarrow \mathfrak{S}$

$$(A_2) \quad \Lambda(\varphi * \alpha) = \hat{\alpha}(s) \Lambda(\varphi) \quad \text{for some (well-chosen) } \alpha \in I_c^\infty(G).$$

$$(A_3) \quad \mathcal{C}\varphi = \lambda_s \varphi.$$

Last time we showed that for $\mathrm{Re}(s) > 1$, $\varphi = E_s$ is the unique solution to $(A) := (A_1) \wedge (A_2) \wedge (A_3)$. This gave the meromorphic continuation. We noted moreover that for each $c'(s)$, $\exists!$ solution φ to $(A_1) \wedge (A_2)$. This allowed us to reduce study of φ to that of c'.

Bernstein–Lapid give a slightly different system characterizing E_s.

$(A_1) \Leftarrow (B_1) \quad \varphi_P = f_{P,s} + c'(s) f_{P,-s}$ for some $c'(s)$

(no restriction to $\mathfrak{S}$)

$(A_2) \Leftarrow (B_2) \quad \varphi * \alpha = \hat{\alpha}(s) \varphi$ $\quad \Big| \ \big((*\alpha) - \hat{\alpha}(s)\big)^N \varphi = 0$ for $N=1$

$(B_3) \quad \varphi \perp L^2_{cusp}$.

$(A_3) \Leftarrow (B_1) \wedge (B_3)$:
$$\left.\begin{array}{l} (B_3) \overset{(*)}{\Rightarrow} \varphi_P \text{ determines } \varphi \\ (B_1) \Rightarrow \mathcal{C}\varphi_P = \lambda_s \varphi_P \end{array}\right\} \Rightarrow \mathcal{C}\varphi = \lambda_s \varphi.$$

$(*)$: for $V := \{\text{smooth uniform moderate growth } \varphi : \Gamma\backslash G \to \mathbb{C}\}$, the map $V \cap (L^2_{cusp})^\perp \ni \varphi \longmapsto \varphi_P$ is injective.
(indeed, if $\varphi_P = 0$, then $\varphi \in V_{cusp} \subseteq L^2_{cusp}$)

(in fact $(B_1) \wedge (B_3)$)

They give a different proof that $(B) \wedge$ characterizes E_s:

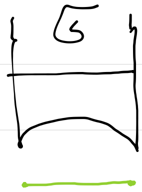
(for $\mathrm{Re}(s)>1$)

Proof: Suppose ψ satisfies $(B_2) \wedge (B_3)$.

Then

$$\psi' := \psi - E_s \text{ satisfies}$$

$$\psi'_P \overset{(B_1)}{=} \left(f_{P,s} - c'(s) f_{P,-s}\right) - \left(f_{P,s} - c(s) f_{P,-s}\right)$$



$$= \underbrace{\left(c(s) - c'(s)\right) f_{P,-s}}_{\text{decays as } y \to \infty \ (\Rightarrow \text{bounded})}$$

$$f_{P,s} = |y|^{1+s}$$

$$\left(\underset{\text{as } y\to\infty}{\psi' \approx \psi'_P}\right) \Longrightarrow \psi' : \text{bounded on } \Gamma \backslash G.$$

$$\Longrightarrow \psi'_P : \text{bounded on } \Gamma_P U \backslash G$$

$$\psi'_P = \left(c(s) - c'(s)\right) \underbrace{f_{P,-s}}_{|y|^{1-s}}, \quad \mathrm{Re}(s) > 1$$

: NOT bounded as $y \to 0$

$$\Longrightarrow c(s) - c'(s) = 0.$$

Thus ψ, E_s are both in $L^2_{\text{cusp}}{}^{\perp}$ and have the same constant terms, hence are equal. $\square$

Langlands's coarse spectral decomposition

$G = SL_n$

Let $\phi : \Gamma \backslash G \to \mathbb{C}$ be locally integrable, moderate growth.

Then $\exists ! \, \phi^{cusp} \in L^2_{cusp}(\Gamma \backslash G)$ s.t. $\forall \, \varphi \in L^2_{cusp}$, $\langle \phi, \varphi \rangle = \langle \phi^{cusp}, \varphi \rangle$.

cuspidal component $\qquad L^2_{cusp}$

of rapid decay after convolving with any $\alpha \in C_c^\infty(G)$

For GL_n, we require instead that $\langle \phi, \varphi \rangle = \langle \phi^{cusp}, \varphi \rangle$ for all φ of the form $\varphi(g) = \Phi(h_G(g)) \, \varphi_0(g)$, where φ_0 : cuspidal automorphic form, $\Phi \in C_c^\infty(\mathbb{R}_+^\times)$, $h_G(g) := |\det g|^{1/n}$.

More generally, $\forall$ standard parabolic P, we define

ϕ_P : constant term, $\quad \phi_P^{cusp} :=$ cuspidal component of ϕ_P.

(in $L^2(\Gamma_P U \backslash G)$)

$$\langle \phi_P, \varphi \rangle = \langle \phi_P^{cusp}, \varphi \rangle \quad \forall \, \varphi(g) = \Phi(h_P(g)) \, \varphi_0(g),$$

φ_0 : cusp form on $\Gamma_P U \backslash G$, $\Phi \in C_c^\infty(A_P)$

$A_P \cong (\mathbb{R}_+^\times)^r$ if $M \cong GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_r}(\mathbb{R})$

Theorem $\phi \neq 0 \implies \phi_P^{cusp} \neq 0 \;\; \exists P$.

(i.e., $\phi_P^{cusp} = 0 \; \forall P \implies \phi = 0$)

Proof We argue by strong induction (on n, or more generally on semisimple rank of G).

(More generally, for any parabolic $Q \leq G$, we have a similar implication concerning $\phi : \Gamma_Q U_Q \backslash G \to \mathbb{C}$.)

By our inductive hypothesis, we may assume that $\forall$ proper parabolic subgroups $P \subsetneq G$, we have $\phi_P = 0$.

By approximating ϕ by $\phi * \alpha$, we may assume ϕ : uniform moderate growth. Then ϕ : cuspidal, hence of rapid decay (possibly modulo the center).

In the SL_n case, we then have $\phi \in L^2_{cusp}$, hence $\phi = \phi^{cusp} = 0$, as required.

The GL_n case is similar (take $\Phi = 1$ on a large interval)

$\leadsto \phi^{cusp} = 0 \implies \phi = 0$. $\square$

Not hard to check: ϕ : automorphic form $\Leftrightarrow$ ϕ_P^{cusp} : automorphic form

Set

$$A_G := \{\text{aut. forms on } \Gamma \backslash G\}.$$

Call two standard parabolic subgroups P, Q associated if $\exists w \in W$ s.t. $w M_P w^{-1} = M_Q$. Notation $P \sim Q$.

For GL_n, the standard parabolics are in bijection with tuples $(n_1, \ldots, n_r)$ s.t. $n_1 + \cdots + n_r = n$. Two such tuples correspond to associated parabolics $\Leftrightarrow$ they are permutations of one another. Thus

$$\{\text{association classes of } P\} \leftrightarrow \{\text{partitions of } n\}$$

For Θ : association class of P's, write

$$A_\Theta := \{\varphi \in A_G : \varphi_P^{cusp} = 0 \ \forall P \notin \Theta\}$$

Then $A_G \cong \bigoplus_\Theta A_\Theta$.

Cuspidal exponents $\phi \in A_G \rightsquigarrow \phi_P^{cusp} \in A_P^{cusp}$,

$$A_P := \{\text{cusp forms on } \Gamma_P U \backslash G\}.$$

Consider the action of A_P on A_P by left translation. We say that $\lambda \in X_P$ is the exponent of $\psi \in A_P$ if $\{a \mapsto a^{\rho+\lambda}\}$ is a generalized eigenvalue for $A_P \curvearrowright \psi$, or equivalently, if $(\forall a \in A_P, g \in G)$

$$\psi(ag) = a^{\rho+\lambda} \sum_{j=1}^{d} P_j(a) \psi_j(g),$$

$$P_j = \text{polynomial in } \log(a).$$

ex $\quad \psi(ag) = a^{\rho+\lambda} \psi(g).$

Defn if $\psi = \sum_{\lambda \in \Lambda} \psi_\lambda$, $\psi_\lambda \neq 0$, $|\Lambda| < \infty$, then $\Lambda =: \{\text{exponents of } \psi\}$.

Defn A cuspidal exponent of $\phi \in A_G$ is a pair (P, λ) s.t. λ : exponent of ϕ_P^{cusp}.

SL_2 $\quad A_G = \{1\}$, $\quad \phi_P = \text{lin. comb. of } f_{P, \pm s}$

$P = \begin{pmatrix} * & * \\ & * \end{pmatrix} \Rightarrow A_P \cong \mathbb{R}_+^\times \quad \Rightarrow \{\text{exponents of } \phi_P\} = \{s, -s\}.$

The Theorem of Langlands stated above gives

$$\mathcal{A}_G \ni \phi \neq 0 \implies \{\text{cuspidal exponents of } \phi\} \neq \emptyset.$$

Bernstein-Lapid prove the following strengthening:

$$\text{———}/\!/\text{———} \implies \left\{ \begin{array}{l} \text{LEADING} \\ \text{cuspidal exponents of } \phi \end{array} \right\} \neq 0,$$

where we say that the exponent $\lambda - \rho_P$ for P is leading if λ: dominant.

(SL_2 case: $f_{P,s} = |y|^{1+s}$: leading exponent $\iff Re(1+s) \geq 0$
$f_{P,-s} = |y|^{1-s}$: ———//——— $\iff Re(s) \leq 1$.)